Commutation and The Uncertainty Principle

The Heisenberg Uncertainty Principle

> *A policeman pulls Werner Heisenberg over on the Autobahn for speeding.*
> *Policeman: Sir, do you know how fast you were going?*
> *Heisenberg: No, but I know exactly where I am.*

I. Definition and Examples

The most general form of the Heisenberg Uncertainty Principle is

$$\Delta A\,\Delta B \;\geq\; \frac{1}{2}\left|\left\langle\left[\hat{A},\ \hat{B}\right]\right\rangle\right| , \qquad (1)$$

where $\hat{A}$ and $\hat{B}$ are Hermitian operators. The uncertainty ΔA in the physical observable A associated with the operator $\hat{A}$ is defined by the relation

$$\Delta A \;=\; \left[\left\langle A^2\right\rangle - \left\langle A\right\rangle^2\right]^{1/2} . \qquad (2)$$

The Heisenberg Uncertainty Principle can be evaluated for specific operators. As an example, consider $\hat{A} = \hat{p}_x$ and $\hat{B} = \hat{x}$. The commutator is $\left[\hat{p}_x,\ \hat{x}\right] = -i\hbar$. Then, the Heisenberg Uncertainty Principle (HUP) takes the form

$$\Delta p_x\, \Delta x \ \geq\ \frac{1}{2}\left|\left\langle\left[\hat{p}_x,\ \hat{x}\right]\right\rangle\right| . \tag{3}$$

The commutator is

$$\left[\hat{p}_x,\ \hat{x}\right] = -i\hbar . \tag{4}$$

Then, the uncertainty principle becomes

$$\begin{aligned} \Delta p_x\, \Delta x &\geq \frac{1}{2}\left|\left\langle\left[\hat{p}_x,\ \hat{x}\right]\right\rangle\right| \\ &= \frac{1}{2}\left|\left\langle \psi \,\middle|\, -i\hbar \,\middle|\, \psi \right\rangle\right| \\ &= \frac{1}{2}\left| -i\hbar \left\langle \psi \,\middle|\, \psi \right\rangle\right| \\ &= \frac{1}{2}\left\{(-i\hbar)^* (-i\hbar)\right\}^{1/2} \\ &= \frac{1}{2}\left\{(i\hbar)(-i\hbar)\right\}^{1/2} \\ \Delta p_x\, \Delta x &\geq \frac{\hbar}{2} . \end{aligned} \tag{5}$$

Variance

- **If ψ is not an eigenfunction of Â, then many values could be obtained if *A* is measured. The average value would be <*Â*>**

$\left\langle \left(\hat{A} - \left\langle \hat{A} \right\rangle\right)^2 \right\rangle$: The standard deviation from the average. The spread of the measured values

$$\left\langle \left(\hat{A} - \left\langle \hat{A} \right\rangle\right)\left(\hat{A} - \left\langle \hat{A} \right\rangle\right) \right\rangle$$

$$= \left\langle \hat{A}^2 - 2\hat{A}\left\langle \hat{A} \right\rangle + \left\langle \hat{A} \right\rangle^2 \right\rangle$$

$$= \left\langle \hat{A}^2 \right\rangle - 2\left\langle \hat{A} \right\rangle\left\langle \hat{A} \right\rangle + \left\langle \hat{A} \right\rangle^2$$

$$= \left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2$$

$$= \sigma_A^{\ 2}$$

Useful integrals for particle in the box

$$\int \sin^2 bx\,dx = \frac{x}{2} - \frac{\sin 2bx}{4b}$$

$$\int x\sin^2 bx\,dx = \frac{x^2}{4} - \frac{x\sin 2bx}{4b} - \frac{\cos 2bx}{8b^2}$$

$$\int x^2 \sin^2 bx\,dx = \frac{x^3}{6} - \left(\frac{x^2}{4b} - \frac{1}{8b^3}\right)\sin 2bx - x\frac{\cos 2bx}{4b^2}$$

⇨ Definite Integrals (Most important). Use $b = \frac{n\pi}{a};\ bx\Big|_{x=a} = \frac{n\pi}{a}a = n\pi$

$$\int_0^a \sin^2 \frac{n\pi x}{a}dx = \frac{a}{2}$$

$$\int_0^a x\sin^2 \frac{n\pi x}{a}dx = \frac{a^2}{4}$$

$$\int_0^a x^2 \sin^2 \frac{n\pi x}{a}dx = \frac{a^3}{6} - \frac{a^3}{4n^2\pi^2}$$

$$\int_0^a \sin\frac{n\pi x}{a}\cos\frac{m\pi x}{a}dx = 0,\ \forall n,m \text{ integers}$$

Demonstration of Uncertainty Principle

Using the above integrals, we can calculate the following

a) Normalize $\psi_n = C_n \sin\frac{n\pi x}{a}$

$$\longrightarrow \quad C_n^{\ 2}\int_0^a \left(\sin\frac{n\pi x}{a}\right)^2 dx = C_n^{\ 2}\,\frac{a}{2} \equiv 1 \qquad C_n = C = \sqrt{\frac{2}{a}}$$

$\Rightarrow$ Normalized particle in the box eigen states: $\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

b) Calculate $\langle x \rangle$ for normalized $\psi_n(x)$:

$$\langle x \rangle = \frac{2}{a}\int_0^a \sin\frac{n\pi x}{a}\cdot x \cdot \sin\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a}\cdot\frac{a^2}{4} = \frac{a}{2} \qquad \text{center of the box}$$

c) Calculate $\langle x^2 \rangle$

$$\langle x^2 \rangle = \frac{2}{a}\int_0^a \sin\frac{n\pi x}{a} \cdot x^2 \cdot \sin\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a}\cdot\left(\frac{a^3}{6} - \frac{a^3}{4n^2\pi^2}\right) = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}$$

d) Standard deviation in $\langle x \rangle$:

$$\sigma_x^{\ 2} = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} - \left(\frac{a}{2}\right)^2 = \frac{a^2}{12} - \frac{a^2}{2n^2\pi^2} = \left(\frac{a}{2\pi n}\right)^2\left[\frac{n^2\pi^2}{3} - 2\right]$$

e) $\left\langle P_x \right\rangle = \frac{2}{a}\int_0^a \sin\frac{n\pi x}{a}\left(-i\hbar\frac{d}{dx}\right)\sin\frac{n\pi x}{a}\,dx$

$$= \frac{2}{a}\left(-i\hbar\frac{n\pi}{a}\right)\int_0^a \sin\frac{n\pi x}{a}\cos\frac{n\pi x}{a}\,dx = 0$$

f) $\left\langle P_x^2 \right\rangle = \frac{2}{a}\int_0^a \sin\frac{n\pi x}{a}\left(-\hbar^2\frac{d^2}{dx^2}\right)\sin\frac{n\pi x}{a}$

$$= \frac{2}{a}\hbar^2 \cdot \frac{n^2\pi^2}{a^2}\int_0^a \left(\sin\frac{n\pi x}{a}\right)^2 dx \rightarrow \frac{a}{2}$$

$$= \frac{\hbar^2 n^2\pi^2}{a^2} = \frac{h^2 n^2}{4a^2}$$ (= $2mE_n$, of course!)

$$\sigma(P_x) = \frac{hn}{2a}$$

We can test the Heisenberg Uncertainty Principle

$$\sigma_x \sigma_p = \frac{a}{2\pi n} \cdot \left[\frac{\pi^2 n^2}{3} - 2 \right]^{\frac{1}{2}} \cdot \frac{hn}{2a}$$

$$= \frac{\hbar}{2}\left[\frac{\pi^2 n^2}{3} - 2 \right]^{\frac{1}{2}} \qquad > \frac{\hbar}{2}$$